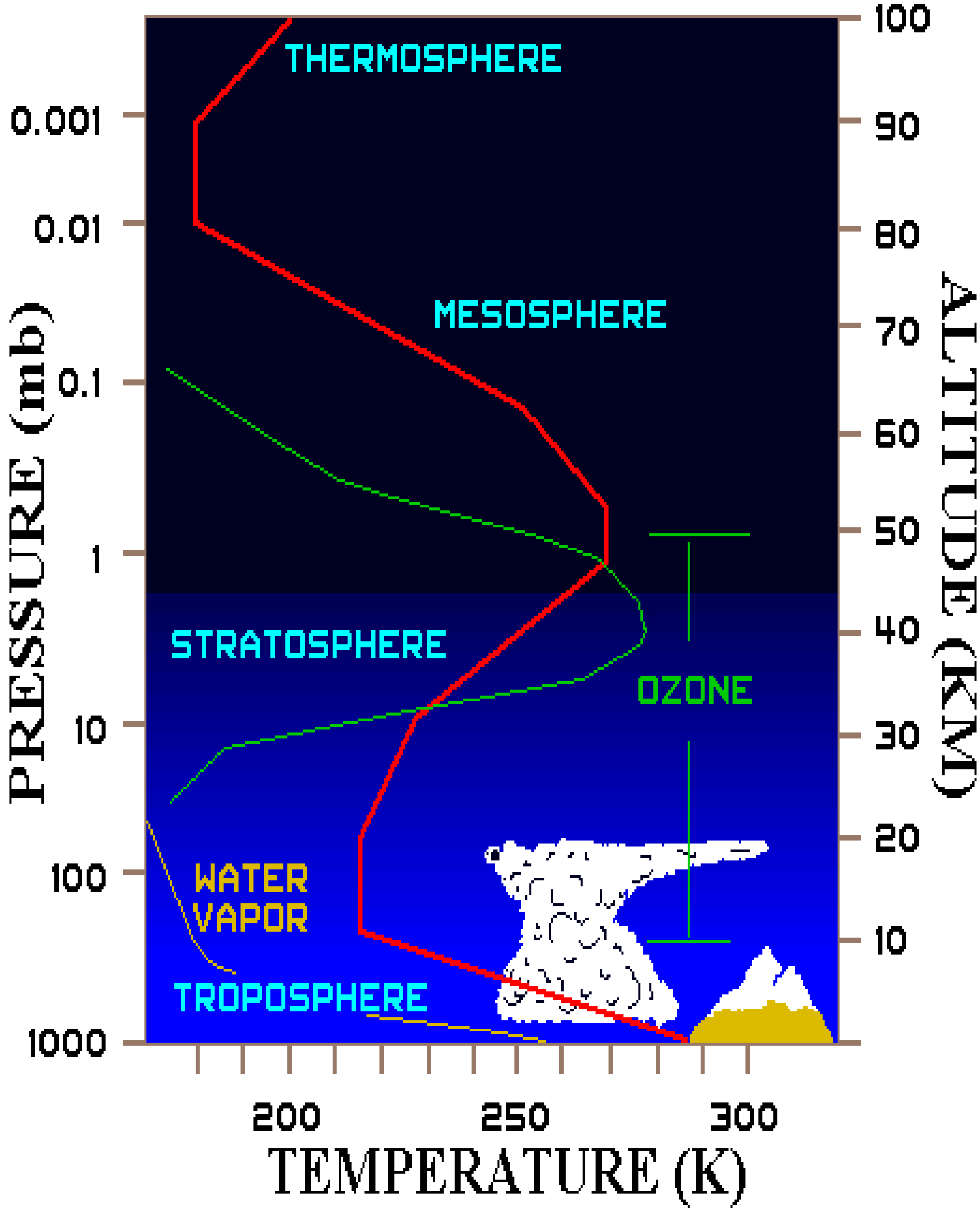
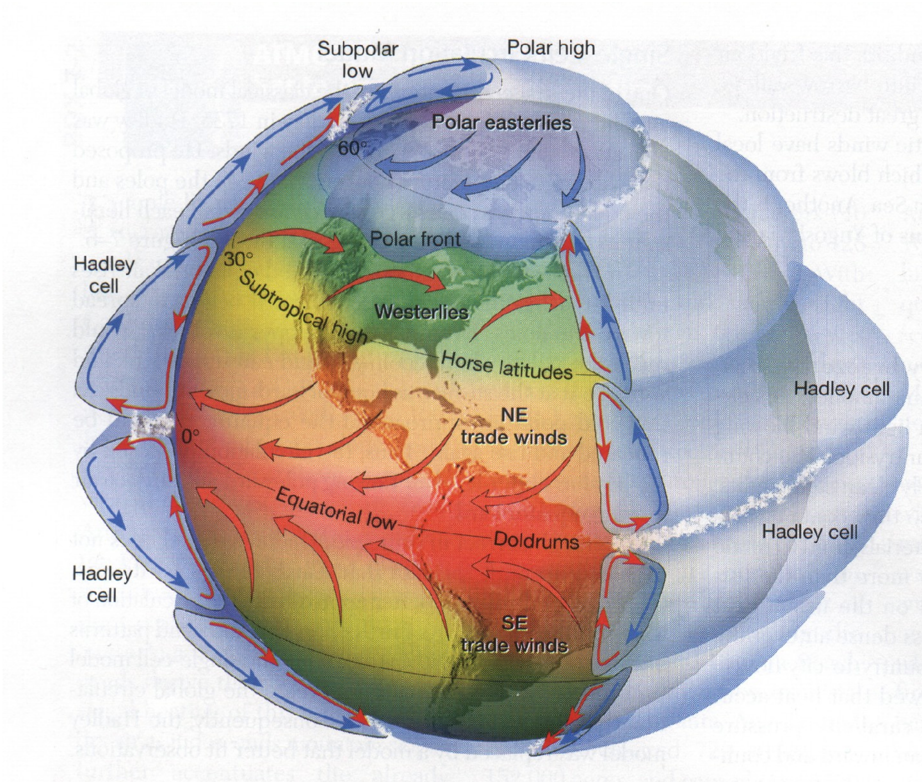
Lecture 14. Mathematical Models Cont'd

1 Structure of the Atmosphere

1.1 Vertical structure of the Atmosphere



1.2 General Circulation



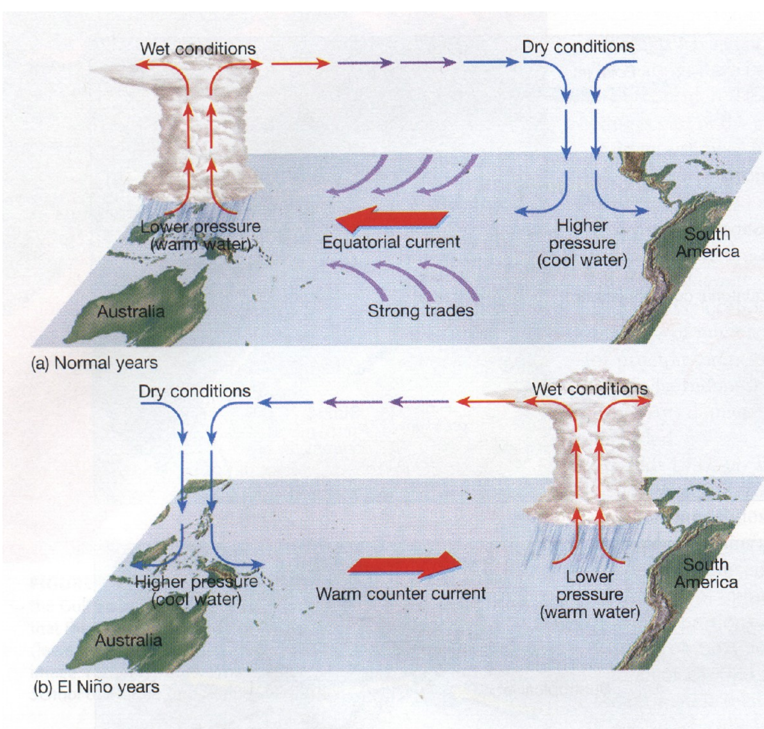


FIGURE 7-24 Simplified illustration of th[e] see-saw pattern of atmospheric pressure between the eastern and western Pacific, called the *Southern Oscillation.* (a) During average years, high pressure over the east[-]ern Pacific causes surface winds and warm equatorial waters to flow westward. The result is a pileup of warm water in the western Pacific, which promotes the lowe[r]ing of pressure. (b) An El Niño event begi[ns] as surface pressure increases in the western Pacific and decreases in the eastern Pacific. This air pressure reversal weakens, or may even reverse the trade winds, and results in an eastward movement of the warm waters that had accumulated in the western Pacific.

Forecasting of El-Nino by GFDL model:
http://www.gfdl.noaa.gov/products/vis/images/gallery/el-nino_la-nina.mov

1.3 Scales

- **Large Scale (Macroscale, global scale, synoptic scale)**
  Trade winds, Rossby, Walker, ENSO, NAO, etc.

- **Medium Scale (Mesoscale, regional scale)**
  Sea-breezes, mountain circulations (foehns, Mistral), etc.
- **Small Scale (Microscale, local scale)**
  Heat islands, internal boundary layers, tornadoes, etc.

2 Primitive Equations

2.1 Equation of State

Equation of state is

$$p = \rho R T_v$$

where p is the pressure [N.m^{-2}], ρ is the air density [kg.m^{-3}], R= 287 [J.kg^{-1}.K^{-1}] is the gas constant, T_v is the virtual temperature [K] given by

$$T_v = (1 + 0.61q)T$$

where T is the absolute temperature [K], and q is the specific humidity. The specific humidity q is expressed as the ratio of mass of water vapor m_w and the mass of moist air m_a, and is related to the mixing ratio of water vapor MR by

$$q = \frac{m_w}{m_a} = \frac{m_w}{m_w + m_d} = \frac{m_w / m_d}{1 + m_w / m_d} = \frac{MR}{1 + MR}$$

NB. The relative humidity is defined as the ratio of the partial pressure of water to its saturation vapor pressure at the same temperature, but as q is generally prognosed in models RH is calculated as $RH = q/q_{sat}(T)$. A formula proposed by Buck (J. App. Meteor., 1527-1532, 1981) is

$$E_w = 6.1121 * (1.0007 + 3.46 \times 10^{-6}\ p) \exp\left(\frac{17.502 \times T}{240.97 + T}\right)$$

$$q_{SAT} = 0.62197\left(\frac{E_w}{p - 0.378 E_w}\right)$$

where p is the pressure [mb], and T the temperature [Celsius]

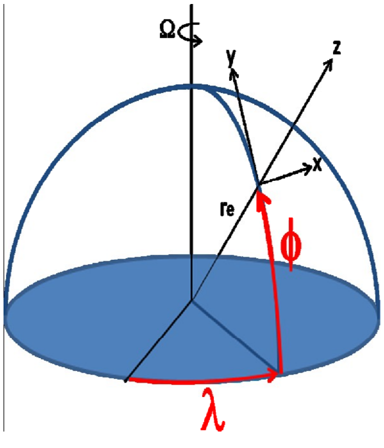
The concentration of air molecules can be calculated from the equation of state. For standard atmospheric conditions (T=298^{o}K and p=1.01325x10^5 [N.m^{-2}]) and using R=8.314 [N.m.mol^{-1}.K^{-1}], the concentration c is given by $c = \frac{p}{RT} = \frac{101325}{8.314 \times 298} = 2.463 \times 10^{19}$ [molec.cm^{-3}]

2.2 Momentum equation

In an inertial reference frame attached to the Earth, Newton's second law of may be written as (cf. Holton, *An introduction to dynamic meteorology*, Academic Press, 1979):

$$\frac{\partial \overline{u}}{\partial t} = -2\overline{\Omega} \times \overline{u} - \frac{1}{\rho}\overline{\nabla} p + \overline{g} + \overline{F_r}$$

Where $\overline{u}$ is the wind vector [m.s^{-1}], $-\frac{1}{\rho}\overline{\nabla} p$ is the pressure gradient force, ρ is the air density [kg.m^{-3}], $\overline{\Omega} \times \overline{u}$ is the Coriolis force, $\overline{\Omega}$ (Ω=7.292x10^{-5}) [sr.s^{-1}] is the vector of the angular velocity of the Earth's rotation, $\overline{g}$ is the effective gravity (centrifugal+gravity), $\overline{F_r}$ is the friction force.



For a system of coordinates (x,y,z) fixed at the Earth's surface, and related to the longitude λ and latitude ϕ, the components of the Coriolis force in the zonal (x), meridional (y), and vertical (z) directions are

$2\Omega v\sin\phi - 2\Omega w\cos\phi$

$-2\Omega u\sin\phi$

$2\Omega u\cos\phi$

By neglecting the second term of the first expression and the third expression, the equations of motion are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+f_c v$$
$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-f_c u$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$

where $f_c = 2\Omega\sin\phi$ is the Coriolis parameter.

Geostrophic approximation considers that the pressure field is balanced by horizontal velocity (which is valid for synoptic scale systems in mid-latitude), and is expressed by

$$-f_c v \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad f_c u \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Hydrostatic approximation considers that the pressure at any point is simply equal to the weight of a unit cross-section of air above that point, and is expressed by

$$dp = -\rho g dz$$

2.3 Thermodynamic equation

The thermodynamic equation is obtained from the total derivative of the equation of state:

$$c_p\frac{d\ln\theta}{dt}=\frac{\dot{q}}{T}$$

Where c_p=1004 [J.kg^{-1}.K^{-1}] is the specific heat at constant pressure, $\theta = T(p_0/p)^{R/c_p}$ is the potential temperature [K], R=287 [J.K^{-1}.kg^{-1}] is the gas constant, p_0=1000 [hPa] is a pressure of reference, T is the absolute temperature [K], and $\dot{q}$ is the rate of heating per unit mass due to radiation, conduction, and release of latent heat.

2.4 Mass conservation

The conservation of air mass is expressed by the continuity equation in its flux form by

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \bar{u}) = 0$$

By using the equality $\nabla \bullet (\rho \bar{u}) = \bar{u} \bullet \nabla \rho + \rho \nabla \bar{u}$, the continuity equation can be expressed by its total differential form

$$\frac{d\rho}{dt} = \frac{\partial}{\partial t}\rho + \bar{u} \bullet \bar{\nabla}\rho = -\rho \nabla \bar{u}$$

2.5 Tracer equation

The mass balance of a tracer *i* is given by the atmospheric diffusion equation where the transport is decomposed into advection and diffusion:

$$\frac{\partial c_i}{\partial t} + \nabla (c_i \bar{u}) = \nabla \bar{\bar{K}} \nabla c_i + (P - L)$$

where $\bar{\bar{K}}$ is the tensor of diffusivity, *P* is the production term, and *L* is the loss term. The equation is generally expressed for aerosols in term of mass mixing ratio $\psi = c_i / \rho_a$ which is equal to the ratio of aerosol concentration c_i and air density ρ_a

2.6 Initial and Boundary conditions

The solution of the partial derivatives equations for momentum, thermodynamics and tracers require the specification of the initial and boundary conditions.

The initial condition (I.C.) consists to specify the three-dimension distribution of all prognostic variables at time *t*=0. If the initial value is unknown, a very low value is generally imposed ("cold-start") and the model is run for a period long enough to have the solution the least influenced by the initial condition. This length of the simulated period influenced by the IC is called spin-up time, and is generally discarded in the analysis of the model results. The final distribution can be saved and used as warm-start initial condition for subsequent simulations. For transport of aerosols, the lifetime in the troposphere is about 2 weeks, which means that the spin-up time should be at least 2 weeks. However, if one wants to study aerosol in the upper troposphere or lower stratosphere, a one year spin-up time is necessary. For General Circulation Model with coupled atmosphere-ocean models, the spin-up time is several hundred years.

The boundary condition (B.C.) consists to specify either the prognostic variable and/or its flux through the boundaries of the domain. For global atmospheric models, there are 2 boundaries: at the model top (generally in the upper stratosphere) and at the Earth's surface. For regional models, there are in addition 4 lateral boundaries.

Lateral boundary conditions

Generally the pragmatic view is taken that if the lateral boundaries are located far enough away from the region of interest, the errors introduced at the boundaries will remain within some acceptable tolerance in the interior of the domain during the simulation period.
There are five types of boundary conditions that can be used:

1. Fixed

These are the simplest boundary conditions that can be applied. All the prognostic variables at the boundaries are specified initially and remain constant with time. These boundary conditions are useful for some theoretical studies.

2. Time-dependent:

The prognostic variable at the boundaries are specified as a smoothly varying function of time and are obtained either from observations, large-scale model simulations, or linear solutions. If the specified values are not consistent with the values near the boundary predicted by the model physics, noise in the simulated variables will develop near the boundaries.

3. Time-dependent and inflow/outflow-dependent

These open boundary conditions allow waves to pass out of the domain. They usually produce smooth solutions.

4. Sponge

The sponge boundary conditions is given by

$$\left(\frac{\partial \alpha}{\partial t}\right)_n = w(n)\left(\frac{\partial \alpha}{\partial t}\right)_{MC} + (1 - w(n))\left(\frac{\partial \alpha}{\partial t}\right)_{LS}$$

where α represents any variable, the subscript MC denotes the model-calculated tendency, an LS the large-scale tendency, which is obtained either from observations or large-scale model simulations, and n is the number of grid points from the nearest boundary (n=1 on the boundary) . The weighting coefficients $w(n)$ vary from 0 to 1 as n increases (from 1 to typically 4).

5. Relaxation

The relaxation boundary condition involves "relaxing" or "nudging" the model-predicted variables toward a large-scale analysis or observations. The method includes a Newtonian and diffusion term

$$\left(\frac{\partial \alpha}{\partial t}\right)_n = F(n)F_1(\alpha_{LS} - \alpha_{MC}) - F(n)F_2\nabla^2(\alpha_{LS} - \alpha_{MC})$$

Where F decreases linearly from the lateral boundary

$$F(n) = \left(\frac{5-n}{3}\right) \quad n = 2,3,4$$

$$F(n) = 0 \qquad n > 4$$

And $F_1 = \dfrac{1}{10\Delta t}$ and $F_2 = \dfrac{\Delta s^2}{50\Delta t}$

Surface boundary conditions

The boundary condition at the surface is generally expressed in terms of turbulent flux. If we separate any variable α into a mean value $\overline{\alpha}$ and a perturbation α'' such that $\alpha = \overline{\alpha} + \alpha''$ and $\overline{\alpha''} = 0$, then the turbulent fluxes at the surface of the prognostic variables are written by the following expressions:

$$(\overline{w''u''})_s = C_m|v_1|$$

$$(\overline{w''v''})_s = C_m|v_1|$$

$$(\overline{w''\theta''})_s = C_h(\theta_s - \theta_1)$$

$$(\overline{w''q''})_s = C_h(q_s - q_1)$$

$$(\overline{w''c''})_s = C_h(c_S - c_1)$$

where C_m and C_h are the exchange coefficient for momentum and heat, respectively. The subscript $_s$ indicates value of variable at the surface and the subscript $_1$ indicates values of variable at the lower model level. $|v_1|, \theta_1, q_1, c_1$ are the wind speed, the potential temperature, the specific humidity, and a tracer (e.g. aerosol) concentration at the lowest model level. The coefficients C_m and C_h are parameterized as a function of the surface roughness z_0, the Richardson number Ri, among others variables. A complete description is given by Stull, *An introduction to boundary layer meteorology*, Kluwer Academic Press, 1988.

SURFACE TEMPERATURE

Over land the surface temperature is computed from a surface energy budget

$$C_g \frac{\partial T_s}{\partial t} = R_n - H_m - H_s - L_v E_s$$

Where C_g is the thermal capacity of the ground per unit area [J.m^{-2}.K^{-1}], R_n the net radiation, H_m the heat flow into the ground, H_s the sensible heat flux into the atmosphere, L_v the latent heat of vaporization, and E_s the surface moisture flux.

1) Net Radiative Flux Rn

Radiation is the driving force of the diabatic planetary boundary layer, and it has two components $R_n = Q_s + I_s$ where Q_s and I_s are the net shortwave and longwave irradiances at the surface.
The amount of solar radiation absorbed by the ground is given by

$$Q_s = t F_0 (1 - R_s) \cos \chi_0$$

where $t = \exp(-\tau) + \omega(1-\beta)(1-\exp(-\tau))$ is the transmitted solar radiation at the surface, F_0=1396 W.m^{-2} the solar constant, R_s the surface albedo, χ_0 is the solar zenith angle (cf. Lecture 4).
The net longwave radiation I_s is equal to the sum of the outgoing ($I \uparrow$) and downward ($I \downarrow$) longwave radiation. The outgoing longwave radiation is

$$I \uparrow = \varepsilon_g \sigma_{SB} T_g^4$$

Where ε_g is the ground emissivity (typical 0.9 to 1), T_g is the ground temperature, σ_{SB}=5.671x10^{-8} W.m^{-2}.K^{-4} is the Stefan-Boltzmann constant. The downward longwave radiation absorbed at the surface is

$$I \downarrow = \varepsilon_g \varepsilon_a \sigma_{SB} T_1^4$$

Where T_1 is the atmospheric temperature at the lowest model level ε_a is the atmospheric longwave emissivity and can be approximated by the relation

$$\varepsilon_a = 0.725 + 0.17 \log_{10} w_p$$

Where w_p is the precipitable water in centimeters.
For cloudy sky, the attenuation of shortwave radiation by cloud is parameterized with absorption and scattering transmissivities.

2) Heat Flow to the ground

The transfer of heat due to molecular conduction is calculated from the equation

$$H_m = C_h C_g (T_g - T_s)$$

Where C_h is the heat-tranfer coefficient, C_g the heat capacity of the ground, T_s is the surface temperature, and T_g the ground temperature.

3) Sensible heat flux

The surface heat flux is given by $H_s = \rho_1 c_{pm} C_h C_m (\theta_g - \theta_1)|v_1|$

4) Surface moisture flux

The surface moisture flux is $E_s = \rho_1 C_h C_m M (q_S(T_g) - q_1)|v_1|$

The sea surface temperature (SST) is generally prescribed based on observations for example the Reynolds and Smith dataset.

SURFACE SPECIFIC HUMIDITY

The specific humidity at the surface can be estimated from the relation

$$q_g = \alpha' q_s(T_g) + (1-\alpha')q_1$$

Where $q_s(T_g)$ is the specific humidity at saturation, q_1 is the specific humidity at the lowest model level, and $\alpha' = \min(1, \frac{w_g}{w_k})$ with w_k the soil porosity and w_g the soil water content. The value of w_g is obtained from the following equation

$$\frac{\partial w_g}{\partial t} = -C_1 \frac{(E_g - P_{rec})}{\rho_w d_g} - C_2 \frac{(w_g - w_2)}{\tau_1}$$

Where E_g is the evaporation rate at the surface, P_{rec} is the precipitation rate, w_g is the groundwater content, and w_2 is the mean value of the groundwater, ρ_w is the water density, C_1=0.1 and C_2=0.9, dg is the maximum depth of diurnal variation, τ_1=86400 s.

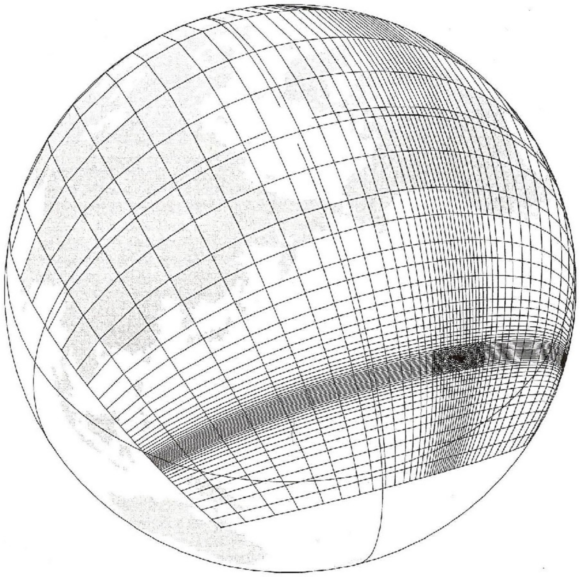
2.7 *Tracer equation in generalized coordinates*

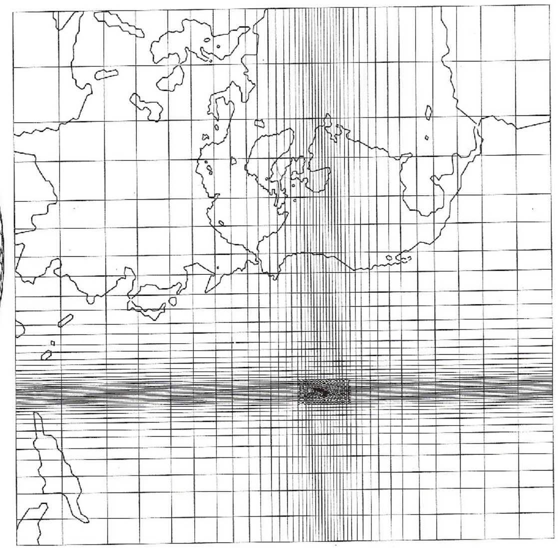
Due to Earth sphericity, atmospheric models are expressed into curvilinear coordinates instead of Cartesian coordinates (e.g. spherical). For regional models, the equations are projected into a Cartesian grid. By using conformal mapping, a simple scaling factor is used which allows to easily go from one coordinate system to the other, once the equation as been expressed in generalized coordinates.

Criteria for selecting a transformation of coordinates:

1. Higher resolution in the studied area (to represent properly all orders of derivatives)
2. Axis of coordinates aligned with flow (to avoid cross derivatives)
3. Move the lateral boundaries far away from the studied area (to avoid contamination of the solution by the B.C.)
4. Simplicity of equations (reduce computing time)

The Following figure shows the stereographic projection of an hemispherical grid which was used to study the effects of long-range transport of pollutants from Asia and North America on the background atmosphere of Hawaii.





The continuity equation of a tracer of mass mixing ratio ψ in curvilinear coordinates is given by

$$\frac{\partial}{\partial t}(G\rho\psi)+\sum_{i=1}^{3}\frac{\partial}{\partial q_i}(G\rho u_i\psi-G\rho(\overline{\overline{K}}\bullet\nabla)_i\psi)=P-L$$

Where ρ is the air density [kg.m-3], $v_i=\frac{dq_i}{dt}$ is the i component of the wind vector, $\overline{\overline{K}}$ is the tensor of diffusion, $G=\det\left(\frac{\partial x_i}{\partial q_j}\right)$ is the Jacobian of the transformation of coordinates from Cartesian x_i to curvilinear q_j

If ψ is the mass mixing ratio of an aerosol, the mass of this aerosol within a volume V is given by

$$M=\int_V G\rho\psi dq_1dq_2dq_3$$

Such that the time variation of M is given by

$$\frac{\partial M}{\partial t}=\int_V\frac{\partial}{\partial t}(G\rho\psi)dq_1dq_2dq_3=-\int_V\sum_{i=1}^{3}\frac{\partial}{\partial q_i}(G\rho v_i\psi-G\rho(\overline{\overline{K}}\bullet\nabla)_i\psi)dq_1dq_2dq_3$$

By applying Stokes' theorem, this last volume integral reduces to the surface integral of the flux of aerosol:

$$\frac{\partial M}{\partial t}=-\overline{n}\bullet\oint(G\rho\overline{v}\psi-G\rho\overline{\overline{K}}\bullet\nabla\psi)$$

In order to transform the coordinates we need to specify the matrix of transformation of coordinates $[T]\equiv\left(\frac{\partial x_i}{\partial q_j}\right)$, then calculate its determinant

$G \equiv \det\left(\frac{\partial x_i}{\partial q_j}\right)$ and its inverse $[T]^{-1} \equiv \left(\frac{\partial q_k}{\partial x_l}\right)$. The components of the wind vector in Cartesian coordinates u_i are related to the curvilinear wind components $\tilde{u}_i = \frac{\partial q_i}{\partial x_k} u_k$, as well as the tensor K_{kl} in Cartesian coordinates is expressed in curvilinear coordinates as $\tilde{K}_{ij} = \sum_{kl} K_{kl} \frac{\partial q_i}{\partial x_k} \frac{\partial q_j}{\partial x_l}$.

Vertical coordinate systems

The vertical coordinate can be space-based (height or depth with respect to a reference surface) or mass-based (pressure, density, potential temperature). Hybrid coordinates with a mass-based element are considered to be mass-based.

The reference surface is a digital elevation map of the planetary surface. This can be a detailed topography or bathymetry digital elevation dataset, or a more idealized one such as the representation of a single simplified mountain or ridge, or none at all. Vertical coordinates requiring a reference surface are referred to as terrain-following (e.g s) and are commonly used.

The pressure-based σ coordinates is defined as

$$\sigma(z) = \frac{p(x,y,z,t) - p_t}{p_s(x,y,t) - p_t} = \frac{p - p_t}{p^*}$$

where p_s and p_t are the surface and top pressures, and p_t is a constant.

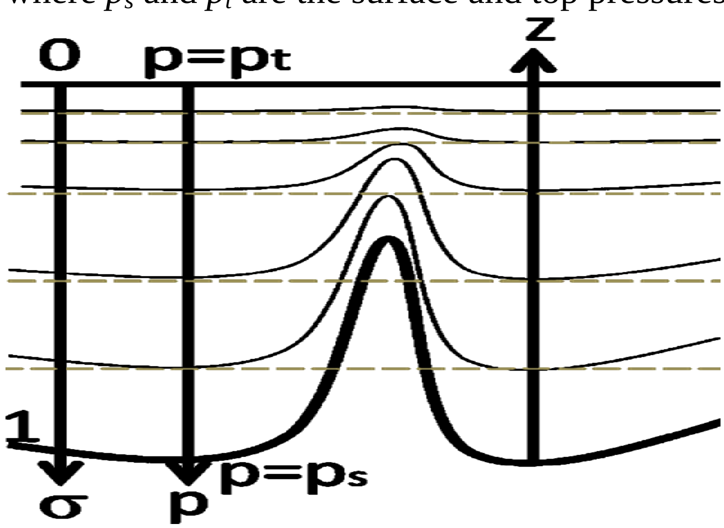


Table 13.1 Conversion scaling factors for vertical coordinate systems.

Coordinate system	m_z	Vertical coordinate
Altitude	1	dz
Pressure	ρg	dp
Sigma $\sigma = (p - p_t)/p^*$	$\rho g / p^*$	$d\sigma$

HYBRID: When the model covers the troposphere, the stratosphere, and eventually the mesosphere, a hybrid system (η) is used instead of sigma (σ). With a η coordinates, near the Earth's surface model levels are defined only by the terrain-following sigma coordinate. In upper model layers, well above the topography, the coordinate surfaces may coincide with constant pressure surfaces. In between there is a slow transition from sigma to pressure. The vertical coordinate is defined by a reference profile of pressure P_0 and sigma/eta values η at the half-model levels. The pressure at half levels can be computed from the surface pressure p_s as $p_{K+\frac{1}{2}} = P_{k+\frac{1}{2}} + \eta_{k+\frac{1}{2}} p_{sl}$.

For sigma coordinates: $p_{sl} = p_s$. For eta coordinates, $p_{sl} = \eta_s p_s$ and 3 conditions should be met: 1) $\eta_{\frac{1}{2}} = 0$; 2) $\eta_{N+\frac{1}{2}} = 1$; 3) $P_{surface} = 0$. If there are N model levels, then pressure is computed at half levels from $p_{\frac{1}{2}}$ to $p_{N+\frac{1}{2}}$. The reference pressures are computed by $P_{k+\frac{1}{2}} = p_{top}(1 - \eta_{k+\frac{1}{2}})$.

Horizontal coordinate systems

Horizontal spatial coordinates may be polar (λ, ϕ) coordinates on the sphere, or planar (x,y), where the underlying geometry is Cartesian, or based on one of several projections of a sphere onto a plane. Planar coordinates based on a spherical projection define a map factor allowing a translation of (x,y) to (λ, ϕ).

Horizontal coordinates may have the important properties of orthogonality (when the Y coordinate is normal to the X) and uniformity (when grid lines in either direction are uniformly spaced). Numerically generated grids may not be able to satisfy both constraints simultaneously.

A third type of horizontal coordinate often used in this domain is not spatial, but spectral. Spectral coordinates on the sphere represent the horizontal distribution of a variable in terms of its spherical harmonic coefficients.

These coefficients can be uniquely mapped back and forth to polar coordinates based on Fourier and Legendre transforms, yielding uniformly spaced longitudes, and latitudes defined by a Gaussian quadrature.

1. Spherical coordinates

We select (*x*, *y*, *z*) as the local Cartesian coordinates, with *x* pointing East, *y* pointing North, and *z* the altitude above sea-level. We select a curvilinear system of coordinates with $q_1 = \lambda$ the longitude (positive to the East), $q_2 = \phi$ the latitude (positive to the North), and $q_3 = \sigma$ defined above.

The Cartesian coordinates are related to the spherical coordinates as follow:

$$
\begin{aligned}
dx &= r_e \cos\phi \, d\lambda \\
dy &= r_e d\phi \\
dz &= -\frac{1}{\rho g} dp \\
&= -\frac{1}{\rho g}\left[(p_s - p_t) d\sigma + \sigma dp_s\right] \\
&= -\frac{p_s - p_t}{\rho g} d\sigma - \frac{\sigma}{\rho g}\frac{\partial p_s}{\partial \lambda} d\lambda - \frac{\sigma}{\rho g}\frac{\partial p_s}{\partial \phi} d\phi - \frac{\sigma}{\rho g}\frac{\partial p_s}{\partial t} dt
\end{aligned}
$$

The matrix [T] is given by

$$
[T] \equiv \begin{pmatrix} r_e \cos\phi & 0 & 0 \\ 0 & r_e & 0 \\ -\dfrac{\sigma}{\rho g}\dfrac{\partial p^*}{\partial \lambda} & -\dfrac{\sigma}{\rho g}\dfrac{\partial p^*}{\partial \phi} & -\dfrac{p^*}{\rho g} \end{pmatrix}
$$

And its inverse is given by

$$
[T]^{-1} \equiv \begin{pmatrix} \dfrac{1}{r_e \cos\phi} & 0 & 0 \\ 0 & \dfrac{1}{r_e} & 0 \\ -\sigma\dfrac{\partial \ln p^*}{\partial x} & -\sigma\dfrac{\partial \ln p^*}{\partial y} & -\dfrac{\rho g}{p^*} \end{pmatrix}
$$

The Jacobian of the transformation is given by

$$
G \equiv \det\left(\frac{\partial x_i}{\partial q_j}\right) = -r_e^2 \frac{p^*}{\rho g}\cos\phi
$$

The wind components ($\dot{\lambda},\dot{\phi},\dot{\sigma}$) in the coordinates system (λ,ϕ,σ) are related to the wind components (u, v, w) by the following relations:

$$\dot{\lambda} = \frac{1}{r_e \cos\phi} u$$

$$\dot{\phi} = \frac{1}{r_e} v$$

$$\dot{\sigma} = -\sigma \frac{\partial \ln p^*}{\partial t} - \sigma \frac{\partial \ln p^*}{\partial x} u - \sigma \frac{\partial \ln p^*}{\partial y} v - \frac{\rho g}{p^*} w$$

Some assumptions are generally made for the tensor of diffusion:

1. It is symmetrical
2. The off-diagonal terms are negligible away from the tropopause and mountainous terrain.

With such assumptions, we get

$$\tilde{K}_{11} = \frac{1}{r_e^2 \cos^2\phi} K_{11}$$

$$\tilde{K}_{22} = \frac{1}{r_e^2} K_{22}$$

$$\tilde{K}_{33} = \sigma^2 \left[\left(\frac{\partial \ln p^*}{\partial x} \right)^2 K_{11} + \left(\frac{\partial \ln p^*}{\partial y} \right)^2 K_{22} \right] + \left(\frac{\rho g}{p^*} \right)^2 K_{33}$$

Therefore, the continuity equation of an aerosol of mass mixing ratio ψ is given by

$$\frac{1}{p^*}\frac{\partial}{\partial t}(p^*\psi) + \frac{1}{p^*}\frac{\partial}{\partial x}(p^* u \psi) + \frac{1}{p^*\cos\phi}\frac{\partial}{\partial y}(p^*\cos\phi\, v\psi) + \frac{\partial}{\partial \sigma}(\dot{\sigma}\psi) = P - L$$
$$+ \frac{1}{p^*}\frac{\partial}{\partial x}(p^* K_{11}\frac{\partial \psi}{\partial x}) + \frac{1}{p^*\cos\phi}\frac{\partial}{\partial y}(p^*\cos\phi\, K_{22}\frac{\partial \psi}{\partial y}) + \frac{\partial}{\partial \sigma}(\left(\frac{\rho g}{p^*}\right)^2 K_{33}\frac{\partial \psi}{\partial \sigma})$$

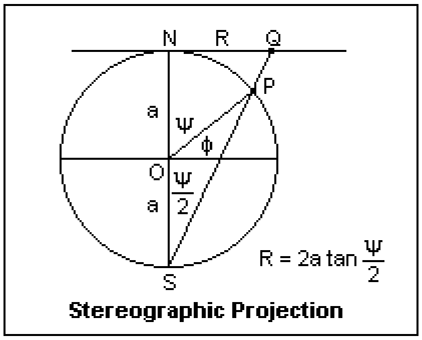
Multiplying this equation by p* and using $K_\sigma \equiv \tilde{K}_{33} = \left(\frac{\rho g}{p^*}\right)^2 K_{33}$, the equation becomes:

$$\frac{\partial}{\partial t}(p^*\psi) + \frac{\partial}{\partial x}(p^* u \psi) + \frac{1}{\cos\phi}\frac{\partial}{\partial y}(p^*\cos\phi\, v\psi) + p^*\frac{\partial}{\partial \sigma}(\dot{\sigma}\psi) = p^*(P - L)$$
$$+ \frac{\partial}{\partial x}(p^* K_{11}\frac{\partial \psi}{\partial x}) + \frac{1}{\cos\phi}\frac{\partial}{\partial y}(p^*\cos\phi\, K_{22}\frac{\partial \psi}{\partial y}) + p^*\frac{\partial}{\partial \sigma}(K_\sigma\frac{\partial \psi}{\partial \sigma})$$

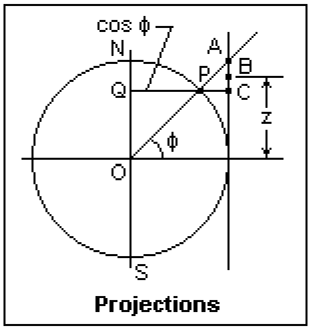
3. Conformal projection

The horizontal scale factor (*m*) is defined as the ratio of the distance on the grid to the corresponding distance on the Earth's surface. The projection is conformal if the scale is equal in all directions about a point so that the shape of geographic features on Earth is preserved. The following projections are preserved and the latitude of intersection between the plane of projection and the Earth is ϕ_1 .

a) Polar stereographic: $m = \dfrac{1+\sin\phi_1}{1+\sin\phi}$

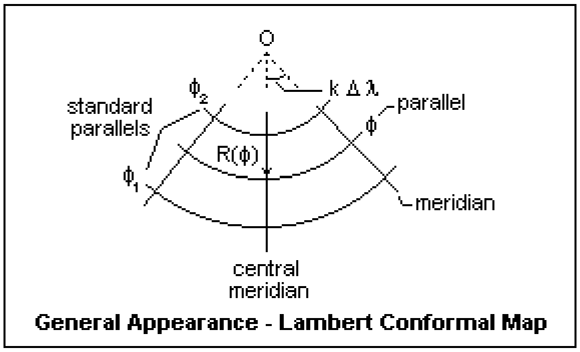


b) Mercator : $m = \dfrac{\cos\phi_1}{\cos\phi}$



c) Lambert conformal: The Lambert conformal grid is true at latitudes 30° and 60°N so that m=1 at these latitudes.

In general, $m = \dfrac{\sin\psi_1}{\sin\phi}\left[\dfrac{\tan\psi/2}{\tan\psi_1/2}\right]^{0.716}$ where ψ_1 =30° and ψ is the colatitudes ($\psi = 90^o - \phi$).



For conformal projection the matrix of projection is given by

$$[T] \equiv \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{-p^*}{\rho g} \end{pmatrix}$$

And the determinant G is given by $G \equiv -\frac{1}{m^2}\frac{p^*}{\rho g}$

The continuity equation for a tracer of mixing ratio ψ is then given by

$$\frac{\partial p^* \psi}{\partial t} = -m^2 \left[\frac{\partial p^* u \psi / m}{\partial x} + \frac{\partial p^* v \psi / m}{\partial y} \right] - \frac{\partial p^* \psi \dot{\sigma}}{\partial \sigma} + p^*(P - L) + D_n$$

Similarly, the momentum and thermodynamic equations are given by (for detailed derivation of the equations see Zdunkowski and Bott, *Dynamics of the Atmosphere: A course in theoretical meteorology*, Cambridge University Press, 2003):

$$\frac{\partial p^* u}{\partial t} = -m^2 \left[\frac{\partial p^* u u / m}{\partial x} + \frac{\partial p^* v u / m}{\partial y} \right] - \frac{\partial p^* u \dot{\sigma}}{\partial \sigma} - m p^* \left[\frac{\sigma}{\rho} \frac{\partial p^*}{\partial x} + \frac{\partial \phi}{\partial x} \right] + p^* f_c v + D_u$$

$$\frac{\partial p^* v}{\partial t} = -m^2 \left[\frac{\partial p^* u v / m}{\partial x} + \frac{\partial p^* v v / m}{\partial y} \right] - \frac{\partial p^* v \dot{\sigma}}{\partial \sigma} - m p^* \left[\frac{\sigma}{\rho} \frac{\partial p^*}{\partial y} + \frac{\partial \phi}{\partial y} \right] - p^* f_c u + D_v$$

$$\frac{\partial p^* T}{\partial t} = -m^2 \left[\frac{\partial p^* u T / m}{\partial x} + \frac{\partial p^* v T / m}{\partial y} \right] - \frac{\partial p^* T \dot{\sigma}}{\partial \sigma} + p^* \frac{w}{\rho c_p} + p^* \frac{\dot{q}}{c_p} + D_T$$

where m is the map scale factor, $p^* = p_s - p_t$, (u,v,w) are the three components of the wind vector, λ is the longitude, ϕ is the latitude, f_c is the Coriolis parameter, $\dot{q}$ is the adiabatic heating, the *D* terms represent the vertical and horizontal diffusion terms and vertical mixing due to the planetary boundary layer or dry convective adjustment, $c_p = c_{pd}(1 + 0.8 q_v)$ where q_v us the mixing ratio for water vapor and c_{pd} is the heat capacity for dry air, and ψ is the mass mixing ratio of tracer. The vertical component of the wind vector is given by

$$w = p^* \dot{\sigma} + \sigma \frac{dp^*}{dt}$$

Surface pressure

$$\frac{\partial p^*}{\partial t} = -m^2 \left[\frac{\partial p^* u / m}{\partial x} + \frac{\partial p^* v / m}{\partial y} \right] - \frac{\partial p^* \dot{\sigma}}{\partial \sigma}$$

which is used in its vertically integrated form

$$\frac{\partial p^*}{\partial t} = -m^2 \int_0^1 \left[\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right] d\sigma$$

and then the vertical velocity in σ-coordinates, $\dot{\sigma}$ is computed by vertical integration

$$\dot{\sigma} = -\frac{1}{p^*} \int_0^\sigma \left[\frac{\partial p^*}{\partial t} + m^2 \left(\frac{\partial p^* u/m}{\partial x} + \frac{\partial p^* v/m}{\partial y} \right) \right] d\sigma$$

The hydrostatic equation is used to compute the geopotential heights from the virtual temperature T_v,

$$\frac{\partial \phi}{\partial \ln(\sigma + p_t / p^*)} = -RT_v \left[1 + \frac{q_c + q_r}{1 + q_v} \right]^{-1}$$

Where $T_v = T(1+0.608q_v)$ is the virtual temperature, and q_c and q_r are the mixing ratios of cloud water and rain water.

Non-hydrostatic Model

For non-hydrostatic model, the pressure, temperature and air density is decomposed into a constant reference state and perturbations, such as

$$p(x,y,z,t) = p_0(z) + p'(x,y,z,t)$$
$$T(x,y,z,t) = T_0(z) + T'(x,y,z,t)$$
$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

With the full pressure at a grid point given by $p = p^*\sigma + p_t + p'$

In general for model grid > 10 km the hydrostatic approximation is valid.

Nudging

The method of Newtonian relaxation or nudging relaxes the model variables toward observations by adding to the prognostic equations, artificial tendency terms based on the difference between the simulated and observed values of the same variable. The model solution can be nudged toward either gridded analyses or individual observations during a period of time surrounding the observations. The predictive equation of variable $\alpha(\mathrm{x},t)$ with a nudging term is given by

$$\frac{\partial \alpha}{\partial t} = F(\alpha, \mathrm{x}, t) + G_\alpha \times \varepsilon_\alpha(\mathrm{x}) \times (\hat{\alpha}_0 - \alpha)$$

Where G_α is the nudging factor which determines the relative magnitude of the term to all the other model processes in F, $\varepsilon_\alpha(\mathrm{x})$ is the observation quality factor (0 to 1), $\hat{\alpha}_0$ is the observation.

The nudging contribution is artificial; therefore it should not be the dominant term in the governing equations. It is scaled by the slowest physical process in the model and must satisfy the numerical stability criterion $G_\alpha \leq \frac{1}{\Delta t}$.

If $\varepsilon_\alpha(\mathrm{x})=1$ then

$$\frac{\partial \alpha}{\partial t} = G_\alpha(\hat{\alpha}_0 - \alpha)$$

.which has the solution $\alpha(t+\Delta t) = \hat{\alpha}_0 + (\alpha(t) - \hat{\alpha}_0)e^{-G_\alpha t}$ where $\alpha(t)$ is the solution at the initial time *t*. The model solution approaches the observation exponentially with an e-folding time of $T_G = 1/G_\alpha$. This implies that high frequency fluctuations in the observations are generally not retained except for high values of G_α; but then the nudging term may not be small compared to *F*.